

## ON OPEN PREFERENCES

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In a paper by Bergstrom, Parks and Rader (1976) some useful facts about open preferences are collected together. In particular, conditions under which open preferences are equivalent to those having open sections are investigated. The purpose of the present note is to complement their paper by presenting some further remarks on open preferences. First, we characterize open preferences by way of a property of sections of preferences. Second, we extend Theorem 3 of Bergstrom, Parks and Rader, which is attributed to Shafer (1974) by them, to the case where the set  $X$  is not necessarily convex. From the first remark we shall present, it is not surprising that conditions guaranteeing the equivalence between open preferences and the ones with open sections are somewhat restrictive. We introduce restrictions in terms of preconvexity of preferences and locally finite preconvexity of consumption sets  $X$ . The second remark is of economic interest especially for general equilibrium models of large economies since recent works by Mas-Colell (1977) and Yamazaki (1976, 1981) have shown that the assumption of convexity of consumption sets are not essential in obtaining standard results in general equilibrium analysis of large economies.

Let  $X$  be a topological space and  $P \subset X \times X$  a relation on  $X$ . ( $X \times X$  is always endowed with the product topology in this paper.) The sets  $P(x) = \{z \in X \mid (x, z) \in P\}$  and  $P^{-1}(x) = \{z \in X \mid (z, x) \in P\}$  for  $x$  in  $X$  are referred to as the *upper* and *lower sections* of  $P$ . If  $P(x)$  and  $P^{-1}(x)$  are open in  $X$  for every  $x$  in  $X$ , then  $P$  is said to have *open sections*.

*Proposition 1: The following conditions are equivalent:*

- (1) *A relation  $P$  is open.*
- (2) *If  $(x, y) \in P$ , then there exists a neighborhood of  $x$ ,  $W(x) \subset P(y)$ , such that  $\bigcap_{z \in W(x)} P^{-1}(z)$  is a neighborhood of  $y$ .*

*Proof.* (1)  $\Rightarrow$  (2): If  $P$  is open, then there exist a neighborhood  $W(x)$  of  $x$  and a neighborhood  $W(y)$  of  $y$  such that  $W(x) \times W(y) \subset P$ .

We claim that  $W(y) \subset \bigcap_{z \in W(x)} P^{-1}(z)$ , thus proving the assertion. Indeed, if  $y' \in W(y)$ , then  $W(x) \times \{y'\} \subset P$ . Thus for all  $z$  in  $W(x)$  one has  $y' \in P^{-1}(z)$ . It follows that  $y' \in \bigcap_{z \in W(x)} P^{-1}(z)$ .

(2)  $\Rightarrow$  (1): Let  $(x, y) \in P$ . Then, there exists a neighborhood of  $x$ ,  $W(x) \subset P(y)$ , such that  $W(y) = \bigcap_{z \in W(x)} P^{-1}(z)$  is a neighborhood of  $y$ . Thus for every  $z$  in  $W(x)$  one has  $W(y) \subset P^{-1}(z)$  or  $\{z\} \times W(y) \subset P$ . It follows that  $W(x) \times W(y) \subset P$ . Q.E.D.

Now, let  $X$  be an arbitrary topological subspace of a real linear topological space. A relation  $P \subset X \times X$  is said to be *preconvex* if  $(\text{co } P(x)) \cap X = P(x)$  for every  $x$  in  $X$ . (If  $H$  is a subset of the linear space,  $\text{co } H$  refers to its convex hull.)  $P$  is preconvex if and only if, for every  $x$  in  $X$ ,  $y, z \in P(x)$ ,  $0 < t < 1$ , and  $ty + (1-t)z \in X$  imply that  $ty + (1-t)z \in P(x)$ . The

concept of preconvex relations is a natural extension of that of convex relations ( $P$  is said to be convex if  $P(x)$  is a convex set for each  $x$  in  $X$ ) to the case where  $X$  is not necessarily a convex set. It is very easy to generate examples of preconvex relations which are not convex: Take a convex relation  $P$  defined on a convex set  $X$ ; then, the restriction of  $P$  to a nonconvex subset  $X'$  of  $X$  is a preconvex relation on  $X'$ . If  $P$  is a preconvex relation on  $X$  and if  $X$  is a convex set, then  $P$  is a convex relation on  $X$ .

A topological subspace  $X$  of a real linear topological space is said to be *locally finitely preconvex* if every  $x$  of  $X$  has a neighborhood base consisting of sets of the form  $(\text{co } F) \cap X$  where  $F \subset X$  is a finite set. We shall exhibit examples of locally finite preconvex subspaces below. These examples show that in a finite dimensional space most commodity spaces, consumption sets and/or sections of a relation which have economic interest belong to this class of subspaces.

*Examples of Locally Finite Preconvex Subspaces:*

- (1)  $R_+^l \subset R^l$ . Existing standard models use  $R_+^l$  as the commodity space.
- (2) Any discrete subset of a real linear topological space. This would be the commodity space and/or consumption sets if all the commodities are indivisible.
- (3) A polytope of a finite dimensional real linear topological space.
- (4)  $Z_+^{l-1} \times R_+ \subset R^l$  where  $Z_+$  represents the set of nonnegative integers. This space is used as the commodity space and consumption sets by Mas-Colell (1977).
- (5) Commodity spaces and consumption sets  $X$  in  $R^l$  which contain mutually exclusive consumption goods [see Yamazaki (1976)]. For example,  $X = \bigcup_{i \in L} R_+^{i-\#} \times \{0\} \times \dots \times R_+ \times \{0\} \times \dots \times \{0\}$  where  $\# \leq l$ .
- (6) Let  $X \subset R^l$  be an arbitrary convex set and  $P \subset X \times X$  has open upper sections. Then  $P(x)$  is locally finitely preconvex in  $X$  for all  $x$  in  $X$ . (One can see this in the following manner: Since  $X$  is a convex set, its dimension is well defined, say it is  $m$ -dimensional; since  $P(x)$  are open in  $X$ , at any point  $x'$  in  $P(x)$   $m$ -dimensional cubes form a neighborhood base at  $x'$ ; corresponding to a cube in this neighborhood base define a finite set  $F$  to be the set of extreme points of the cube; the collection of finite sets  $F$  thus defined has the desired property.)

We now establish the following results:

*Proposition 2:*

- (1) Let  $X$  be a locally finite preconvex subspace of a real linear topological space, and  $P \subset X \times X$  a preconvex relation on  $X$  having open sections. Then,  $P$  satisfies the property (2) of Proposition 1.
- (2) Let  $X$  be a topological subspace of a real linear topological space and  $P \subset X \times X$  a preconvex relation on  $X$  having open sections. Then, if  $P$  has locally finite preconvex upper sections,  $P$  satisfies the property (2) of Proposition 1.

*Corollary 1:* Let  $X$  be a locally finite preconvex subspace of a real linear topological space and  $P \subset X \times X$  a preconvex relation on  $X$ . Then,  $P$  is open if and only if  $P$  has open sections.

*Corollary 2:* Let  $X$  be a topological subspace of a real linear topological space and  $P \subset X \times X$  a preconvex relation on  $X$  having locally finite preconvex upper sections. Then,  $P$  is open if and only if  $P$  has open sections.

The above two corollaries are the restatement of *Proposition 2*. The following result is an immediate consequence of Corollary 2 and the remark in (6) of the previous examples.

*Corollary 3:* Let  $X \subset R^l$  be an arbitrary convex subset and  $P \subset X \times X$  a convex relation. Then,  $P$  is open if and only if  $P$  has open sections.

*Proof of Proposition 2.*

(1) Let  $(x, y)$  be in  $P$ . Since  $P$  has open sections and  $X$  is locally finitely preconvex, there is a finite subset  $F$  of  $X$  such that  $W(x) = (\text{co } F) \cap X$  is a neighborhood of  $x$  and  $W(x) \subset P(y)$ . Thus one has  $y \in P^{-1}(z)$  for all  $z$  in  $W(x) \supset F$ . Set  $W(y) = \bigcap_{z \in F} P^{-1}(z)$ .  $W(y)$  is a neighborhood of  $y$  because  $y$  is in  $W(y)$  and  $W(y)$  is a finite intersection of open sets  $P^{-1}(z)$ . We shall show that  $W(y) \subset \bigcap_{z \in W(x)} P^{-1}(z)$ . Let  $y' \in W(y)$ ; then  $(z, y') \in P$  for all  $z$  in  $F$ , i.e.,  $F \subset P(y')$ . Hence it follows from the preconvexity of the relation  $P$  that  $(\text{co } F) \cap X \subset (\text{co } P(y')) \cap X = P(y')$ . Thus one has  $W(x) \subset P(y')$ . It implies that  $y' \in P^{-1}(z)$  for all  $z$  in  $W(x)$ .

(2) The proof follows from (1) by noting that it is enough to have locally finite preconvex upper sections of  $P$  in generating a required finite subset  $F$  in the above proof.

*Q.E.D.*

*Remark:* Bergstrom et al. (1976, Theorem 3, p. 267) showed that if  $X = R^l_+$  and  $P$  is convex, then  $P$  is open if and only if  $P$  has open sections. Corollary 3 above extends their result to an arbitrary convex consumption set. In fact, in view of the remark (6) of the above examples, Proposition 2 extends their Theorem 3 to the case where  $P$  is not a convex relation.

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